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Parameter shifts, D_4 symmetry and joint eigenfunctions for commuting Askey–Wilson-type difference operators

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Abstract
In previous papers we studied a generalized hypergeometric function that is a joint eigenfunction of four hyperbolic Askey–Wilson difference operators. Using parameter shifts related to the D_4 weight lattice, we show that this function is the even linear combination of two elementary joint eigenfunctions for a dense subset of the parameter space; more specifically, the latter eigenfunctions are equal to a plane wave multiplied by hyperbolic functions.

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1. Introduction

In this paper we obtain various explicit results concerning joint eigenfunctions of commuting second-order analytic difference operators with quite special hyperbolic coefficients. In order to put these results in a more general context, it is, however, illuminating to begin by considering an extensive class of second-order analytic difference operators (henceforth AΔOs), with coefficients from the space $\mathcal{M}$ of meromorphic functions.

To this end we first define the translation operator

$$(T_\eta F)(z) \equiv F(z-\eta) \qquad \eta \in \mathbb{C}^* \qquad F \in \mathcal{M} \tag{1.1}$$

and fix numbers

$$a_+, a_- \in (0, \infty). \tag{1.2}$$

Now we introduce the AΔO

$$A_+ \equiv C_+(z)T_{\mathrm{i}a_-} + C_+(-z)T_{-\mathrm{i}a_-} + C_+^{(0)}(z) \tag{1.3}$$

where $C_+(z)$ and $C_+^{(0)}(z)$ are $\mathrm{i}a_+$-periodic functions in $\mathcal{M}$. We view A_+ as a linear operator $\mathcal{M} \to \mathcal{M}$ throughout this paper. As such, it clearly commutes with any AΔO of the form

$$A_- \equiv C_-(z)T_{\mathrm{i}a_+} + C_-(-z)T_{-\mathrm{i}a_+} + C_-^{(0)}(z) \tag{1.4}$$

where $C_-(z)$ and $C_-^{(0)}(z)$ are ia_--periodic. Therefore, it is natural to ask whether joint eigenfunctions exist. To be specific, for given $E_+, E_- \in \mathbb{C}$, the question is whether functions $F \in \mathcal{M}^*$ exist solving the second-order AΔ Es (analytic difference equations)

$$A_+ F = E_+ F \qquad A_- F = E_- F. \tag{1.5}$$

There appears to be virtually no literature on this type of problem. Focusing at first on the eigenvalue problem for A_+, it seems not even the existence of any eigenfunction $F \in \mathcal{M}^*$ is known. When we *assume* $F \in \mathcal{M}^*$ satisfies $A_+ F = E_+ F$, it is, however, plain that the solution space is infinite dimensional. Indeed, we can multiply F by any function μ in $\mathcal{P}_{\mathrm{i}a_-}$, where

$$\mathcal{P}_\eta \equiv \{\mu \in \mathcal{M} \mid \mu(z+\eta) = \mu(z)\} \qquad \eta \in \mathbb{C}^*. \tag{1.6}$$

Clearly, for two solutions $F_1, F_2 \in \mathcal{M}^*$, the Casorati determinant

$$\mathcal{D}(F_1, F_2; z) \equiv F_1(z + \mathrm{i}a_-/2) F_2(z - \mathrm{i}a_-/2) - F_1(z - \mathrm{i}a_-/2) F_2(z + \mathrm{i}a_-/2) \tag{1.7}$$

vanishes identically if $F_1(z) = \mu(z) F_2(z)$ with $\mu \in \mathcal{P}_{\mathrm{i}a_-}$. Assuming $\mathcal{D}(F_1, F_2; z) \in \mathcal{M}^*$, it is also not hard to see that the solution space is two dimensional over the field $\mathcal{P}_{\mathrm{i}a_-}$, with basis functions F_1, F_2 [1]. But it should be repeated that this result pertains to *given* solutions $F_1, F_2 \in \mathcal{M}^*$—we are not aware of an existence result.

Returning to the joint eigenfunction problem pertinent to the present paper, there is a crucial dichotomy governed by the quotient $a_+/a_- \in (0, \infty)$. Whenever it is rational, the existence of a joint solution $F \in \mathcal{M}^*$ to (1.5) entails infinite dimensionality of the solution space. But since we have

$$\mathcal{P}_{\mathrm{i}a_+} \cap \mathcal{P}_{\mathrm{i}a_-} = \mathbb{C} \qquad a_+/a_- \notin \mathbb{Q} \tag{1.8}$$

it is very likely that the joint solution space can be at most two dimensional for irrational a_+/a_-.

A complete proof of this expectation can be readily obtained under an additional assumption on given joint solutions $F^{(\pm)} \in \mathcal{M}^*$. Specifically, it suffices to assume

$$\lim_{\mathrm{Im} z \to \infty} F^{(+)}(z)/F^{(-)}(z) = 0 \qquad \mathrm{Re}\, z \in [a, b] \tag{1.9}$$

cf section 1 in [2], especially equations (1.13)–(1.16). (The AΔOs occurring in [2] are slightly different, but it is easy to adapt the reasoning.)

The latter two-dimensionality result plays a pivotal role in this paper. It was obtained first in a special context where two joint eigenfunctions satisfying (1.9) do exist, cf theorem B.1 in [3]. In the latter setting the allowed coefficient functions are elliptic, whereas for our present purposes the results in [4] on the hyperbolic specialization are relevant. In that case the AΔOs A_+ (1.3) and A_- (1.4) are explicitly given by

$$A_\delta(b) = \frac{\sinh(\pi(z - \mathrm{i}b)/a_\delta)}{\sinh(\pi z/a_\delta)} T_{\mathrm{i}a_{-\delta}} + (\mathrm{i} \to -\mathrm{i}) \qquad \delta = +, - \qquad b \in \mathbb{C}. \tag{1.10}$$

(In particular, $C_\pm^{(0)}(z) = 0$.)

We proceed to summarize some results pertaining to the 1-parameter family of commuting hyperbolic AΔOs $A_\pm(b)$. This will facilitate an understanding of the more elaborate results for the 4-parameter (Askey–Wilson type) family of commuting hyperbolic AΔOs on which we focus in this paper (defined in (1.29) and (1.30)). Furthermore, the 1-parameter ('A_1') family is related to the 4-parameter ('BC_1') family in more than one way. It is beyond our present scope to elaborate on this issue, but we intend to return to it elsewhere.

To begin with, joint $A_\pm(b)$-eigenfunctions are only known to exist when the eigenvalues $(E_+, E_-) \in \mathbb{C}^2$ lie on a curve of the form

$$E_\pm = 2\cosh(a_\mp y) \qquad y \in \mathbb{C}. \tag{1.11}$$

(Note that for $a_+ = a_-$ the AΔOs $A_+(b)$ and $A_-(b)$ coincide, so in that case (1.11) is necessary.) The simplest case is the 'free' case $b = 0$, where $A_\pm(b)$ reduce to

$$T_{\mathrm{i}a_\mp} + T_{-\mathrm{i}a_\mp}. \tag{1.12}$$

Then the plane waves $\exp(\pm \mathrm{i}zy)$ are obviously joint eigenfunctions with eigenvalues (1.11). However, even for $b = 0$ it seems not a straightforward task to obtain or rule out joint eigenfunctions for an eigenvalue pair not on the curve (1.11). Anyway, in the remainder of this paper we restrict attention to eigenvalues E_+ and E_- related by (1.11).

Next, we introduce the AΔO

$$\mathcal{S}_+^{(u)} \equiv \frac{-\mathrm{i}}{2\sinh(\pi z/a_+)}(T_{\mathrm{i}a_-} - T_{-\mathrm{i}a_-}). \tag{1.13}$$

A simple calculation shows that it shifts the parameter in $A_+(b)$ up by a_-

$$\mathcal{S}_+^{(u)} A_+(b) = A_+(b + a_-)\mathcal{S}_+^{(u)}. \tag{1.14}$$

(One need only use the relation

$$\sinh(x) + \sinh(y) = 2\sinh((x+y)/2)\cosh((x-y)/2) \tag{1.15}$$

to check (1.14).) It is also clear by inspection that $\mathcal{S}_+^{(u)}$ satisfies

$$\mathcal{S}_+^{(u)} A_-(b) = A_-(b + a_-)\mathcal{S}_+^{(u)}. \tag{1.16}$$

The functions

$$\mathcal{S}_+^{(u)N} \exp(\pm \mathrm{i}zy) \qquad N \in \mathbb{N} \tag{1.17}$$

are therefore joint eigenfunctions of $A_\pm(Na_-)$ with eigenvalues (1.11).

The eigenfunctions (1.17) were already presented in equations (3.38)–(3.41) of our survey [5]. Though we obtained the latter via the shift (1.13), we did not spell this out in [5]. In a somewhat different guise, they were studied in [6, 7]. In [4] we used them as a building block to construct joint eigenfunctions for $A_\pm(N_+a_+ + N_-a_-)$ with $N_+, N_- \in \mathbb{Z}$. From our present perspective, however, it is expedient to obtain joint eigenfunctions for the latter b-values via further parameter shifts.

Specifically, the AΔO

$$\mathcal{S}_-^{(u)} \equiv \frac{-\mathrm{i}}{2\sinh(\pi z/a_-)}\left(T_{\mathrm{i}a_+} - T_{-\mathrm{i}a_+}\right) \tag{1.18}$$

satisfies

$$\mathcal{S}_-^{(u)} A_\delta(b) = A_\delta(b + a_+)\mathcal{S}_-^{(u)} \qquad \delta = +, -. \tag{1.19}$$

(One need only interchange a_+ and a_- in (1.14) and (1.16) to check this.) Note that $\mathcal{S}_-^{(u)}$ and $\mathcal{S}_+^{(u)}$ commute. Setting

$$\mathcal{F}_{k,l}^{(\pm)}(z, y) \equiv \mathcal{S}_+^{(u)k}\mathcal{S}_-^{(u)l} \exp(\pm \mathrm{i}zy) \qquad k, l \in \mathbb{N} \tag{1.20}$$

we now get joint eigenfunctions of $A_\pm(ka_- + la_+)$ for k, l non-negative integers.

To obtain eigenfunctions for negative integers, too, we introduce the commuting AΔO pair

$$\mathcal{S}_\delta^{(d)}(b) \equiv \frac{2\mathrm{i}}{\sinh(\pi z/a_\delta)}[\sinh(\pi(z - \mathrm{i}b)/a_\delta)\sinh(\pi(z + \mathrm{i}a_{-\delta} - \mathrm{i}b)/a_\delta)T_{\mathrm{i}a_{-\delta}} - (\mathrm{i} \to -\mathrm{i})]. \tag{1.21}$$

These AΔOs shift the parameter b down by $a_{-\delta}$

$$\mathcal{S}_\delta^{(d)}(b)A_{\delta'}(b) = A_{\delta'}(b - a_{-\delta})\mathcal{S}_\delta^{(d)}(b). \tag{1.22}$$

Indeed, just as for the up-shifts, relations (1.22) are immediate for $\delta' = -\delta$, whereas for $\delta' = \delta$ one need only use (1.15). As a consequence, $\mathcal{S}_\delta^{(u)}\mathcal{S}_\delta^{(d)}(b)$ and $\mathcal{S}_\delta^{(d)}(b + a_{-\delta})\mathcal{S}_\delta^{(u)}$ commute with $A_\pm(b)$. This is in accordance with the identities

$$\mathcal{S}_\delta^{(d)}(b + a_{-\delta})\mathcal{S}_\delta^{(u)} = A_\delta(b)^2 - 4c_\delta^2(\mathrm{i}b) \qquad \mathcal{S}_\delta^{(u)}\mathcal{S}_\delta^{(d)}(b) = A_\delta(b)^2 - 4c_\delta^2(\mathrm{i}b - \mathrm{i}a_{-\delta}) \tag{1.23}$$

whose verification is once more a straightforward calculation using (1.15).

Combining the above shifts, we can now obtain joint eigenfunctions for all parameters in the set

$$\{(a_+, a_-, N_-a_- + N_+a_+) \mid a_+, a_- > 0, N_-, N_+ \in \mathbb{Z}\} \tag{1.24}$$

which is clearly dense in $(0, \infty)^2 \times \mathbb{R}$. Specifically, letting again $k, l \in \mathbb{N}$, we get in addition to (1.20) joint eigenfunctions

$$\mathcal{F}^{(\pm)}_{-k,-l}(z, y) \equiv \prod_{m=0}^{k-1} \mathcal{S}_+^{(d)}(-ma_- - la_+) \cdot \prod_{n=0}^{l-1} \mathcal{S}_-^{(d)}(-na_+) \cdot \exp(\pm \mathrm{i}zy) \tag{1.25}$$

$$\mathcal{F}^{(\pm)}_{-k,l}(z, y) \equiv \prod_{m=0}^{k-1} \mathcal{S}_+^{(d)}(-ma_- + la_+) \cdot \mathcal{S}_-^{(u)l} \exp(\pm \mathrm{i}zy) \tag{1.26}$$

$$\mathcal{F}^{(\pm)}_{k,-l}(z, y) \equiv \mathcal{S}_+^{(u)k} \prod_{n=0}^{l-1} \mathcal{S}_-^{(d)}(-na_+) \cdot \exp(\pm \mathrm{i}zy) \tag{1.27}$$

of the AΔOs $A_\pm(-ka_- - la_+)$, $A_\pm(-ka_- + la_+)$ and $A_\pm(ka_- - la_+)$, respectively.

We now turn to the definition of the commuting hyperbolic AΔOs of Askey–Wilson type [8, 9]. Employing henceforth the notation

$$s_\delta(z) \equiv \sinh(\pi z/a_\delta) \qquad c_\delta(z) \equiv \cosh(\pi z/a_\delta) \qquad e_\delta(z) \equiv \exp(\pi z/a_\delta) \qquad \delta = +, - \tag{1.28}$$

we define

$$A_\delta(\mathbf{c}; z) \equiv C_\delta(\mathbf{c}; z)\big(T_{\mathrm{i}a_{-\delta}} - 1\big) + C_\delta(\mathbf{c}; -z)\big(T_{-\mathrm{i}a_{-\delta}} - 1\big) + 2c_\delta(\mathrm{i}(c_0 + c_1 + c_2 + c_3)) \qquad \delta = +, - \tag{1.29}$$

where

$$C_\delta(\mathbf{c}; z) \equiv \frac{s_\delta(z - \mathrm{i}c_0)}{s_\delta(z)} \frac{c_\delta(z - \mathrm{i}c_1)}{c_\delta(z)} \frac{s_\delta(z - \mathrm{i}c_2 - \mathrm{i}a_{-\delta}/2)}{s_\delta(z - \mathrm{i}a_{-\delta}/2)} \frac{c_\delta(z - \mathrm{i}c_3 - \mathrm{i}a_{-\delta}/2)}{c_\delta(z - \mathrm{i}a_{-\delta}/2)}. \tag{1.30}$$

Clearly, $A_+(\mathbf{c}; z)$ and $A_-(\mathbf{c}; z)$ are of the general form (1.3) and (1.4) discussed above. It is also obvious that the AΔOs $A_\pm((b, 0, 0, 0); z)$ are equal to $A_\pm(b)$ (1.10).

Introducing a new parameter vector γ by

$$\gamma_0 \equiv c_0 - (a_+ + a_-)/2 \qquad \gamma_1 \equiv c_1 - a_-/2 \qquad \gamma_2 \equiv c_2 - a_+/2 \qquad \gamma_3 \equiv c_3 \tag{1.31}$$

we see that (1.30) entails

$$C_\delta(\mathbf{c}(\gamma); z) = -\frac{\prod_{\mu=0}^{3} 2c_\delta(z - \mathrm{i}\gamma_\mu - \mathrm{i}a_{-\delta}/2)}{4s_\delta(2z)s_\delta(2z - \mathrm{i}a_{-\delta})} \equiv V_\delta(\gamma; z) \qquad \delta = +, -. \tag{1.32}$$

The AΔOs $A_\delta(\mathbf{c}(\gamma); z)$ are therefore invariant under arbitrary permutations of $\gamma_0, \ldots, \gamma_3$. In section 2 we introduce 16 γ-shifts that play the same role for the 4-parameter family (1.29) as the above four shifts $\mathcal{S}_\pm^{(u)}, \mathcal{S}_\pm^{(d)}$ for the 1-parameter family (1.10). (They are given by (2.2) and (2.11)–(2.13).) More specifically, starting from the zero coupling case

$$\gamma_f \equiv (-(a_+ + a_-)/2, -a_-/2, -a_+/2, 0) \Rightarrow \mathbf{c}(\gamma_f) = 0 \tag{1.33}$$

where the AΔOs (1.29) reduce to (1.12), we can construct joint eigenfunctions for a dense subset of the parameter space

$$\Pi \equiv \{(a_+, a_-, \gamma) \in \mathbb{R}^6 \mid a_+, a_- > 0\}. \tag{1.34}$$

At this point we should mention that some of the γ-shifts were first presented by Chalykh in his comprehensive study of (mostly multi-variable) Baker–Akhiezer-type eigenfunctions [10]. More specifically, he focuses on entire eigenfunctions, and in a one-variable 'trigonometric' Askey–Wilson setting he introduces shifts that amount to the shifts $S_+^{(-r_\mu)}$ given by (2.11) and (2.12).

Section 2 is basically self-contained. The calculations for the BC_1 family (1.29) are far more extensive than for the A_1 family (1.10), but they still have an elementary character. As will become clear in section 3, the joint eigenfunctions constructed in section 2 are of an auxiliary nature, but they play a pivotal role. Section 3 is not self-contained, inasmuch as we need substantial input from our papers [11, 12], which we refer to as I and II from now on.

We proceed to collect the pertinent information. The 'relativistic' hypergeometric function $R(a_+, a_-, \mathbf{c}; v, \hat{v})$ studied in I and II is a joint eigenfunction of

$$A_+(\mathbf{c}; v) \qquad A_-(I\mathbf{c}; v) \qquad A_+(\hat{\mathbf{c}}; \hat{v}) \qquad A_-(I\hat{\mathbf{c}}; \hat{v}) \tag{1.35}$$

with eigenvalues

$$2c_+(2\hat{v}) \qquad 2c_-(2\hat{v}) \qquad 2c_+(2v) \qquad 2c_-(2v). \tag{1.36}$$

Here, I is the transposition

$$I\mathbf{c} \equiv (c_0, c_2, c_1, c_3) \tag{1.37}$$

and the dual couplings $\hat{\mathbf{c}}$ are given by $J\mathbf{c}$, with

$$J \equiv \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{1.38}$$

In the present context it is in fact more convenient to work with a renormalized R-function R_r. It differs from the function R_{ren} occurring in I and II only in its dependence on the parameters. Specifically, we set

$$R_r(a_+, a_-, \gamma; v, \hat{v}) \equiv R_{\mathrm{ren}}(a_+, a_-, \mathbf{c}(\gamma); v, \hat{v}). \tag{1.39}$$

The function R_r is real-analytic on Π (1.34) and meromorphic in v and $\hat{v}$; poles can only occur for

$$\pm v = -\mathrm{i}\gamma_\mu + z_{kl} \qquad \pm\hat{v} = -\mathrm{i}\hat{\gamma}_\mu + z_{kl} \qquad \mu \in \{0, 1, 2, 3\} \qquad k, l \in \mathbb{N} \tag{1.40}$$

$$z_{kl} \equiv \mathrm{i}(k + 1/2)a_+ + \mathrm{i}(l + 1/2)a_- \tag{1.41}$$

and their maximal multiplicity is known, cf I theorem 2.2. It differs from the R-function by a proportionality factor that only depends on the parameters. (Since this factor has poles and zeros independent of v and $\hat{v}$, it is inconvenient for our present purposes; if we would work with R, it would also give rise to clutter in various formulae occurring below.)

Symmetry properties that are easily derived from the definition of the R-function (for which we refer to I) include 'modular invariance'

$$R_r(a_+, a_-, \gamma; v, \hat{v}) = R_r(a_-, a_+, \gamma; v, \hat{v}) \tag{1.42}$$

self-duality

$$R_r(a_+, a_-, \gamma; v, \hat{v}) = R_r(a_+, a_-, \hat{\gamma}; \hat{v}, v) \tag{1.43}$$

scale invariance

$$R_r(a_+, a_-, \gamma; v, \hat{v}) = R_r(\lambda a_+, \lambda a_-, \lambda\gamma; \lambda v, \lambda\hat{v}) \qquad \lambda > 0 \tag{1.44}$$

and evenness

$$R_r(a_+, a_-, \gamma; v, \hat{v}) = R_r(a_+, a_-, \gamma; \delta v, \delta'\hat{v}) \qquad \delta, \delta' = +, -. \tag{1.45}$$

A symmetry feature that is much harder to obtain is D_4 symmetry. (In the appendix we have collected some well-known facts concerning the Lie algebra D_4 and its Weyl group W.) This refers to a function $\mathcal{E}$ obtained from R_r by similarity transforming with the c-function

$$c(p; z) \equiv \frac{1}{G(a_+, a_-; 2z + \mathrm{i}(a_+ + a_-)/2)} \prod_{\mu=0}^{3} G(a_+, a_-; z - \mathrm{i}p_\mu). \tag{1.46}$$

(The G-function is the hyperbolic gamma function from [13].) Specifically, we have

$$\mathcal{E}(\gamma; v, \hat{v}) \equiv \chi(\gamma) R_r(\gamma; v, \hat{v}) / c(\gamma; v) c(\hat{\gamma}; \hat{v}) \tag{1.47}$$

with χ a D_4 invariant phase factor, cf II (1.18) and (1.19). The D_4 invariance of $\mathcal{E}$ (see II theorem 1.1) entails that R_r satisfies

$$\frac{R_r(\gamma^{(1)}; v, \hat{v})}{R_r(\gamma^{(2)}; v, \hat{v})} = \frac{c(\gamma^{(1)}; v) c(J\gamma^{(1)}; \hat{v})}{c(\gamma^{(2)}; v) c(J\gamma^{(2)}; \hat{v})} \qquad \gamma^{(j)} \equiv w_j(\gamma) \qquad w_j \in W \qquad j = 1, 2 \tag{1.48}$$

a relation we have occasion to invoke below.

Basically, the reason why $\mathcal{E}$ is D_4 invariant is the D_4 invariance of the similarity transformed AΔOs. To be sure, this symmetry is not manifest. Indeed, no similarity transformation can change the 'additive potential'

$$V_{b,\delta}(\gamma; z) \equiv -V_\delta(\gamma; z) - V_\delta(\gamma; -z) - 2c_\delta\left(\mathrm{i}\left(\sum_{\mu=0}^{3} \gamma_\mu + a_{-\delta}\right)\right) \tag{1.49}$$

in $A_\delta(\mathbf{c}(\gamma); z)$. The D_4 invariance of this function is not clear by inspection, but it follows from II lemma 2.1.

Another result from II that is essential for our purposes concerns the $\mathrm{Re}\, v \to \infty$ asymptotics of the $\mathcal{E}$-function, which we need only for $a_+ \neq a_-$. Setting

$$\mathcal{E}_{\mathrm{as}}(a_+, a_-, \gamma; v, \hat{v}) \equiv \exp(\mathrm{i}\alpha v\hat{v}) + \frac{c(a_+, a_-, \hat{\gamma}; -\hat{v})}{c(a_+, a_-, \hat{\gamma}; \hat{v})} \exp(-\mathrm{i}\alpha v\hat{v}) \qquad \alpha \equiv 2\pi/a_+a_- \tag{1.50}$$

we have from II theorem 1.2

$$\mathcal{E}(\gamma; v, \hat{v}) - \mathcal{E}_{\mathrm{as}}(\gamma; v, \hat{v}) = O(\mathrm{e}^{-\rho \mathrm{Re} v}) \qquad \mathrm{Re}\, v \to \infty \tag{1.51}$$

with $\rho > 0$ and the bound uniform for $(\gamma, \mathrm{Im}\, v, \hat{v})$ in compact subsets of $\mathbb{R}^4 \times \mathbb{R} \times (0, \infty)$. Using the c-function asymptotics II (3.5), this entails

$$R_r(\gamma; v, \hat{v}) = \exp(-\alpha v(\hat{\gamma}_0 + (a_+ + a_-)/2))[R_{r,\mathrm{as}}(\gamma; v, \hat{v}) + O(\mathrm{e}^{-\rho \mathrm{Re} v})] \qquad \mathrm{Re}\, v \to \infty \tag{1.52}$$

with

$$R_{r,\mathrm{as}}(\gamma; v, \hat{v}) \equiv c(\hat{\gamma}; \hat{v}) \exp(\mathrm{i}\alpha v\hat{v}) + (\hat{v} \to -\hat{v}). \tag{1.53}$$

To appreciate the role of the asymptotic behaviour (1.52), (1.53) in section 3, we now explain how it entails the crucial identity

$$R_r(\gamma_f; v, \hat{v}) = \exp(\mathrm{i}\alpha v\hat{v}) + \exp(-\mathrm{i}\alpha v\hat{v}). \tag{1.54}$$

Since the AΔOs $A_\pm$ reduce to the free ones (1.12) for $\gamma = \gamma_f$ (recall (1.33)), the plane waves on the rhs of (1.54) are joint eigenfunctions with eigenvalues $c_\pm(2\hat{v})$. The latter clearly satisfy (1.9) with $z \to v$ and $[a, b] \subset (0, \infty)$, so for $a_+/a_- \notin \mathbb{Q}$ they span the space of joint eigenfunctions. Thus $R_r(\gamma; v, \hat{v})$ is a linear combination of the plane waves with (*a priori*) $\hat{v}$-dependent coefficients. Recalling R_r is even in v, we obtain (1.54) up to a proportionality factor $p(\hat{v})$. Now from the duplication formula for the hyperbolic gamma function [13]

$$G(a_+, a_-; 2z) = \prod_{\delta,\delta'=+,-} G(a_+, a_-; z + \mathrm{i}(\delta a_+ + \delta' a_-)/4) \tag{1.55}$$

we obtain $c(\gamma_f; z) = 1$, so from (1.52) and (1.53) we readily deduce $p(\hat{v}) = 1$, hence (1.54).

In section 3 we generalize this reasoning in order to tie in R_r with the elementary joint eigenfunctions obtained in section 2 via the shifts. This allows us to deduce that R_r itself satisfies simple shift relations, namely (3.11), (3.13), (3.15) and (3.16). We would like to point out that it appears quite intractable to obtain these relations directly from the integral representation for R_r (given by equations (1.30)–(1.35) in I with $\prod_j G(\mathrm{i}s_j)$ replaced by 1). In fact, when the shift relations are expressed in terms of the latter representation, they yield identities that look utterly unlikely.

Once we have proved the shift formulae for R_r, we can combine them with (1.54) to conclude that R_r is an elementary function for all γ in the set $\gamma_f + a_- P_- + a_+ P_+$, where $P_\pm$ are copies of the D_4 weight lattice P, cf the appendix. Using (1.48) we then show that the same is true for parameters in the union of the W-transforms of this set, namely,

$$\Pi_{\mathrm{el}} \equiv \{(a_+, a_-, \gamma) \in \Pi \mid \gamma = w(\gamma_f) + a_-\lambda_- + a_+\lambda_+, w \in W, \lambda_\pm \in P\}. \tag{1.56}$$

Furthermore, from (1.47) we deduce that the $\mathcal{E}$-function is elementary on Π_{el}, too.

Thus far, we have used the term ‘elementary’ in a casual way. For our later purposes, however, it is expedient to be more precise: we use this term for functions of the form

$$\sum_{\sigma=+,-} \rho^{(\sigma)}(e_+(v), e_-(v), e_+(\hat{v}), e_-(\hat{v})) \exp(\mathrm{i}\sigma\alpha v\hat{v}) \tag{1.57}$$

where $\rho^{(\pm)}$ are rational functions of their four arguments. We abbreviate the latter property by saying that $\rho^{(\pm)}$ are ‘hyperbolic’. It is easy to see that whenever a function admits a representation (1.57) with $\rho^{(\pm)}$ hyperbolic, then it is unique. Equivalently, when a function of the form (1.57) vanishes identically, then $\rho^{(+)}$ and $\rho^{(-)}$ vanish identically.

Denoting the two summands of R_r and $\mathcal{E}$ for parameters in Π_{el} by $R_r^{(\sigma)}$ and $\mathcal{E}^{(\sigma)}$, we therefore obtain a great many properties of the elementary functions

$$R_r^{(\pm)}(a_+, a_-, \gamma; v, \hat{v}) \qquad \mathcal{E}^{(\pm)}(a_+, a_-, \gamma; v, \hat{v}) \qquad (a_+, a_-, \gamma) \in \Pi_{\mathrm{el}} \tag{1.58}$$

which are inherited from R_r and $\mathcal{E}$, respectively. For instance, the functions $R_r^{(\pm)}$ satisfy (1.42)–(1.44) and they are joint eigenfunctions of the same four AΔOs as R_r.

2. Parameter shifts: first steps

The weights of the even spinor representation of the D_4 Lie algebra are of the form

$$\pm r_\mu \qquad r_\mu \equiv \mathrm{Row}(J)_\mu \qquad \mu = 0, 1, 2, 3 \tag{2.1}$$

where J is the matrix (1.38). (Indeed, J maps the weights $\pm e_\mu$, $\mu = 0, 1, 2, 3$ of the defining D_4 representation to those of the even spinor representation, cf the appendix.) In this section we introduce eight pairs of parameter shifts, each pair being associated with one of the weights $\pm r_\mu$.

The simplest pair corresponds to $r_0 = (1, 1, 1, 1)/2$: it is given by

$$S_\delta^{(r_0)}(z) \equiv \frac{-\mathrm{i}}{2s_\delta(2z)}\left(T_{\mathrm{i}a_{-\delta}/2} - T_{-\mathrm{i}a_{-\delta}/2}\right) \qquad \delta = +, -. \tag{2.2}$$

We claim that the following shift relations hold:

$$S_\delta^{(r_0)}(z)A_{\delta'}(\mathrm{c}; z) = A_{\delta'}(\mathbf{c} + a_{-\delta}r_0; z)S_\delta^{(r_0)}(z) \qquad \delta, \delta' = +, -. \tag{2.3}$$

By $a_+ \leftrightarrow a_-$ symmetry we need only prove this for $\delta' = +$. Taking first $\delta = +$, it suffices to show that when we shift the translations to the right, then the coefficients of $T_{3\mathrm{i}a_-/2}$ and $T_{\mathrm{i}a_-/2}$ on the rhs and lhs are equal. (Indeed, by evenness in z, this entails equality of the coefficients of $T_{-3\mathrm{i}a_-/2}$ and $T_{-\mathrm{i}a_-/2}$.)

Using (1.29) and (1.32), it is immediate that the coefficients of the largest shift $T_{3\mathrm{i}a_-/2}$ are equal, so we turn to those of $T_{\mathrm{i}a_-/2}$. On the rhs we have a coefficient (cf (1.31))

$$\left[-V_+(\gamma + a_-r_0; z) - V_+(\gamma + a_-r_0; -z) + 2c_+\left(\mathrm{i}\left(\sum_{\mu=0}^{3}\gamma_\mu + a_+ + 3a_-\right)\right)\right]\frac{-\mathrm{i}}{2s_+(2z)}$$
$$+ V_+(\gamma + a_-r_0; z)\frac{\mathrm{i}}{2s_+(2z - 2\mathrm{i}a_-)} \tag{2.4}$$

whereas on the lhs we obtain

$$\frac{-\mathrm{i}}{2s_+(2z)}\left[-V_+(\gamma; z - \mathrm{i}a_-/2) - V_+(\gamma; -z + \mathrm{i}a_-/2) + 2c_+\left(\mathrm{i}\left(\sum_{\mu=0}^{3}\gamma_\mu + a_+ + a_-\right)\right)\right]$$
$$+ \frac{\mathrm{i}}{2s_+(2z)}V_+(\gamma; z + \mathrm{i}a_-/2). \tag{2.5}$$

Omitting the common factor $-\mathrm{i}/2s_+(2z)$, we should therefore show equality of

$$\frac{4\prod c_+(z - \mathrm{i}\gamma_\mu - \mathrm{i}a_-)}{s_+(2z - \mathrm{i}a_-)s_+(2z - 2\mathrm{i}a_-)} + \frac{4\prod c_+(z - \mathrm{i}\gamma_\mu - \mathrm{i}a_-)}{s_+(2z)s_+(2z - \mathrm{i}a_-)} + \frac{4\prod c_+(z + \mathrm{i}\gamma_\mu + \mathrm{i}a_-)}{s_+(2z)s_+(2z + \mathrm{i}a_-)}$$
$$- 2c_+(\mathrm{i}(g + 3a_-)) \qquad g \equiv \sum_{\mu=0}^{3}\gamma_\mu \tag{2.6}$$

and

$$\frac{4\prod c_+(z - \mathrm{i}\gamma_\mu)}{s_+(2z + \mathrm{i}a_-)s_+(2z)} + \frac{4\prod c_+(z - \mathrm{i}\gamma_\mu - \mathrm{i}a_-)}{s_+(2z - \mathrm{i}a_-)s_+(2z - 2\mathrm{i}a_-)} + \frac{4\prod c_+(z + \mathrm{i}\gamma_\mu)}{s_+(2z - \mathrm{i}a_-)s_+(2z)} - 2c_+(\mathrm{i}(g + a_-)). \tag{2.7}$$

To this end we note that (2.6) and (2.7) are both $\mathrm{i}a_+$-periodic meromorphic functions of z with finite limits for $\mathrm{Re}\, z \to \pm\infty$. These limits are easily checked to be equal. By Liouville's theorem, equality of (2.6) and (2.7) will therefore follow when the residues at the (generically) simple poles $z = 0, \pm\mathrm{i}a_-/2, \mathrm{i}a_-$ and $z = \mathrm{i}a_+/2, \pm\mathrm{i}a_-/2 + \mathrm{i}a_+/2, \mathrm{i}a_- + \mathrm{i}a_+/2$ are equal. It is routine to verify this, and so the shift relation (2.3) is now proved for $\delta = \delta' = +$.

To demonstrate (2.3) for $\delta' = +$ and $\delta = -$, we begin by noting that the factor $1/s_-(2z)$ in $S_-^{(r_0)}(z)$ commutes with $A_+(\mathbf{c}; z)$ and that the constant terms in $A_+(\mathbf{c}; z)$ and $A_+(\mathbf{c} + a_+r_0; z)$ are equal. Thus we need only check equality of

$$\left(T_{\mathrm{i}a_+/2} - T_{-\mathrm{i}a_+/2}\right)[V_+(\gamma; z)(T_{\mathrm{i}a_-} - 1) + V_+(\gamma; -z)(T_{-\mathrm{i}a_-} - 1)] \tag{2.8}$$

and

$$[V_+(\gamma + a_+r_0; z)(T_{\mathrm{i}a_-} - 1) + V_+(\gamma + a_+r_0; -z)(T_{-\mathrm{i}a_-} - 1)](T_{\mathrm{i}a_+/2} - T_{-\mathrm{i}a_+/2}). \tag{2.9}$$

This amounts to

$$V_+(\gamma; z - \mathrm{i}a_+/2) = V_+(\gamma + a_+ r_0; z) \tag{2.10}$$

and three similar relations, whose validity is plain from (1.32).

We proceed to define parameter shifts for $-r_0$ and $\pm r_k$, $k = 1, 2, 3$. In contrast to the shifts $S_\delta^{(r_0)}(z)$ (2.2), they also depend on γ. Specifically, we need

$$S_\delta^{(-r_0)}(\gamma; z) \equiv \frac{-\mathrm{i}}{2s_\delta(2z)} \left(\prod_{\mu=0}^{3} 2c_\delta(z - \mathrm{i}\gamma_\mu) \cdot T_{\mathrm{i}a_{-\delta}/2} - \prod_{\mu=0}^{3} 2c_\delta(z + \mathrm{i}\gamma_\mu) \cdot T_{-\mathrm{i}a_{-\delta}/2} \right) \tag{2.11}$$

$$S_\delta^{(-r_k)}(\gamma; z) \equiv \frac{-\mathrm{i}}{2s_\delta(2z)} \left(4c_\delta(z - \mathrm{i}\gamma_0) c_\delta(z - \mathrm{i}\gamma_k) T_{\mathrm{i}a_{-\delta}/2} - (\mathrm{i} \to -\mathrm{i}) \right) \qquad k = 1, 2, 3 \tag{2.12}$$

$$S_\delta^{(r_k)}(\gamma; z) \equiv \frac{-\mathrm{i}}{2s_\delta(2z)} \left(4c_\delta(z - \mathrm{i}\gamma_l) c_\delta(z - \mathrm{i}\gamma_m) T_{\mathrm{i}a_{-\delta}/2} - (\mathrm{i} \to -\mathrm{i}) \right) \tag{2.13}$$

where $\{k, l, m\} = \{1, 2, 3\}$. As the generalization of (2.3), we now have 32 shift relations

$$S_\delta^{(\epsilon r_\mu)}(\gamma; z) A_{\delta'}(\mathbf{c}(\gamma); z) = A_{\delta'}(\mathbf{c}(\gamma) + \epsilon a_{-\delta} r_\mu; z) S_\delta^{(\epsilon r_\mu)}(\gamma; z) \tag{2.14}$$

where $\epsilon, \delta, \delta' = +, -$ and $\mu = 0, 1, 2, 3$. Just as for (2.3), these are quite easily verified for $\delta = -\delta'$, whereas for $\delta = \delta'$ their verification proceeds along the same lines as detailed above for (2.3). (Note that by permutation invariance one need only check (2.14) for one of the six pairs (2.12), (2.13).)

From (2.14) it is clear that we have

$$\left[S_\delta^{(-\epsilon r_\mu)}(\gamma + \epsilon a_{-\delta} r_\mu; z) S_\delta^{(\epsilon r_\mu)}(\gamma; z), A_{\delta'}(\mathbf{c}(\gamma); z) \right] = 0 \tag{2.15}$$

where $[\cdot, \cdot]$ denotes the commutator and $\delta, \delta', \epsilon = +, -$. This is consistent with the identities

$$S_\delta^{(-\epsilon r_\mu)}(\gamma + \epsilon a_{-\delta} r_\mu; z) S_\delta^{(\epsilon r_\mu)}(\gamma; z) = A_\delta(\mathbf{c}(\gamma); z) + 2c_\delta(2\mathrm{i}\hat{\gamma}_\mu + \mathrm{i}\epsilon a_{-\delta}). \tag{2.16}$$

To explain why these identities hold true, we first point out it is immediate that the coefficients of $T_{\pm \mathrm{i}a_{-\delta}}$ on the lhs and rhs are equal. Therefore, one need only check equality of the functions that remain. As before, this can be achieved by comparing poles and asymptotics, and then using Liouville's theorem. (More quickly, equality follows from the D_4 invariance of function (1.49).)

Next, we assert that all of the shift commutators except the ones following from (2.16) vanish. To be specific, we have

$$\begin{aligned} S_{\sigma'}^{(-\epsilon' r_{\mu'})}(\gamma + \epsilon a_{-\sigma} r_\mu; z) S_\sigma^{(\epsilon r_\mu)}(\gamma; z) - S_\sigma^{(\epsilon r_\mu)}(\gamma - \epsilon' a_{-\sigma'} r_{\mu'}; z) S_{\sigma'}^{(-\epsilon' r_{\mu'})}(\gamma; z) \\ = 4\epsilon \delta_{\sigma\sigma'} \delta_{\mu\mu'} \delta_{\epsilon\epsilon'} s_\sigma(2\mathrm{i}\hat{\gamma}_\mu) s_\sigma(\mathrm{i}a_{-\sigma}) \end{aligned} \tag{2.17}$$

where $\sigma, \sigma', \epsilon, \epsilon' = +, -$ and $\mu, \mu' = 0, 1, 2, 3$. This assertion can be readily verified by exploiting the following identity:

$$\begin{aligned} &\frac{1}{s_\delta(2z - 2u)} [c_\delta(z + v - u) c_\delta(z + w - u) - c_\delta(z - v - u) c_\delta(z - w - u)] \\ &\quad = \frac{1}{s_\delta(2z + 2u)} [c_\delta(z + v + u) c_\delta(z + w + u) - c_\delta(z - v + u) c_\delta(z - w + u)]. \end{aligned} \tag{2.18}$$

(To prove (2.18), one can either use the above Liouville reasoning or invoke well-known hyperbolic addition formulae.)

Let us now start from the choice $\gamma = \gamma_f$ (cf (1.33)). Then the AΔOs reduce to the free ones (1.12), so that the plane waves $\exp(\pm \mathrm{i}zy)$ are joint eigenfunctions. Clearly, we can now use the shifts in a stepwise fashion to obtain joint eigenfunctions for any γ of the form

$$\gamma(M, N) \equiv \gamma_f + \sum_{\nu=0}^{3}(M_\nu a_- + N_\nu a_+)r_\nu \qquad M, N \in \mathbb{Z}^4. \tag{2.19}$$

(For example, when $M_\mu > 0$, we use $S_+^{(r_\mu)}$ M_μ times, and when $M_\mu < 0$ we use $S_+^{(-r_\mu)}$ $-M_\mu$ times.) Due to the shift commutativity just established, the path along which we arrive at $\gamma(M, N)$ is immaterial. More precisely, setting

$$|M| \equiv \sum_{\nu=0}^{3}|M_\nu| \qquad |N| \equiv \sum_{\nu=0}^{3}|N_\nu| \tag{2.20}$$

we can allow any path with the minimal step number

$$L \equiv |M| + |N|. \tag{2.21}$$

(Equivalently, we should not 'backtrack'.)

The joint eigenfunctions of $A_\pm(\mathbf{c}(\gamma); z)$ obtained in this way will be denoted $F_{M,N}^{(\pm)}(z, y)$. Even though formulae for these functions analogous to (1.20), (1.25)–(1.27) can be written down, they are very unwieldy and we will not do so. Instead, we finish this section by deriving some features of the eigenfunctions that are of decisive importance in the following section.

First, we note that all of the shifts commute with the parity operator

$$(\mathcal{P}F)(z) \equiv F(-z) \qquad F \in \mathcal{M}. \tag{2.22}$$

Since we have

$$F_{0,0}^{(\pm)}(z, y) = \exp(\pm \mathrm{i}zy) \tag{2.23}$$

we deduce recursively the relations

$$F_{M,N}^{(+)}(-z, y) = F_{M,N}^{(-)}(z, y) \qquad M, N \in \mathbb{Z}^4. \tag{2.24}$$

This entails in particular that when one of the functions vanishes identically for some $y = y_0$, so does the other one.

Secondly, we elucidate the structure of the coefficient functions

$$\mathcal{C}_{M,N}^{(\pm)}(z, y) \equiv F_{M,N}^{(\pm)}(z, y)\exp(\mp \mathrm{i}zy). \tag{2.25}$$

Each of the shifts in the L-fold product acting on the plane waves $\exp(\pm \mathrm{i}zy)$ has two terms, cf (2.2) and (2.11)–(2.13). Multiplying out, we see that $\mathcal{C}_{M,N}^{(\delta)}(z, y)$ equals a sum of 2^L terms of the form

$$C(y)H(z)/S(z) \tag{2.26}$$

where S, H and C are given by

$$S(z) = \prod_{m=1}^{|M|} 2s_+(2z + \mathrm{i}r_{+,m}) \cdot \prod_{n=1}^{|N|} 2s_-(2z + \mathrm{i}r_{-,n}) \qquad r_{+,m}, r_{-,n} \in \mathbb{R} \tag{2.27}$$

$$H(z) = \prod_{k=1}^{K} 2c_{\delta_k}(z + \mathrm{i}r_k) \qquad \delta_k \in \{+, -\} \qquad r_k \in \mathbb{R} \tag{2.28}$$

$$C(y) = \chi \exp[(d_- a_- + d_+ a_+)y/2] \tag{2.29}$$

with

$$\chi \in \{1, \mathrm{i}, -1, -\mathrm{i}\} \qquad d_-, d_+ \in \mathbb{Z} \qquad |d_-| \leqslant |M| \qquad |d_+| \leqslant |N|. \tag{2.30}$$

Clearly, the integer K in (2.28) satisfies $0 \leqslant K \leqslant 4L$, and it is the same for all of the 2^L terms. Likewise, the number of δ_k in (2.28) equal to + and − is the same.

Thirdly, we study asymptotic properties. According to the previous paragraph, the $\mathrm{Re}\, z \to \infty$ asymptotics of each term is of the form

$$\mathrm{e}^{\mathrm{i}\phi} C(y)\, \mathrm{e}^{\rho z}[1 + O(\exp(-\pi z/a_l))] \qquad a_l \equiv \max(a_+, a_-) \qquad \mathrm{Re}\, z \to \infty \tag{2.31}$$

where the rate ρ is the same for all terms and $\mathrm{e}^{\mathrm{i}\phi}$ denotes a phase (which does vary from term to term, of course). Now the $\mathrm{Re}\, y \to \infty$ asymptotics of $C(y)$ is plain from (2.29); in particular, we see that the unique term with $d_- = |M|$ and $d_+ = |N|$ diverges faster than all other ones. This entails that when we choose $\mathrm{Re}\, z$, $\mathrm{Re}\, y \geqslant \Lambda_{M,N} > 0$, with $\Lambda_{M,N}$ sufficiently large, then the functions $\mathcal{C}^{(\pm)}_{M,N}(z, y)$ stay at a finite distance from 0. Thus we have $F^{(\pm)}_{M,N}(z, y) \in \mathcal{M}^*$ for all y in the half plane $\mathrm{Re}\, y \geqslant \Lambda_{M,N}$.

Next, we observe that from (2.27) and (2.28) it follows that for $\mathrm{Re}\, z > 0$ fixed, the function $H(z)/S(z)$ stays bounded as $\mathrm{Im} z \to \infty$. Thus the same is true for $\mathcal{C}^{(\pm)}_{M,N}(z, y)$. Fixing y in the half plane $\mathrm{Re}\, y \geqslant \Lambda_{M,N}$ and $\mathrm{Re}\, z$ in some subinterval $[a, b]$ of $[\Lambda_{M,N}, \infty)$, we deduce from the above that we have

$$\lim_{\mathrm{Im} z \to \infty} \frac{F^{(+)}_{M,N}(z, y)}{F^{(-)}_{M,N}(z, y)} = \lim_{\mathrm{Im} z \to \infty} \frac{\mathcal{C}^{(+)}_{M,N}(z, y)}{\mathcal{C}^{(-)}_{M,N}(z, y)}\, \mathrm{e}^{2\mathrm{i}zy} = 0. \tag{2.32}$$

Consequently, fixing $a_+/a_- \notin \mathbb{Q}$ and $M, N \in \mathbb{Z}^4$, the space of all joint $A_\pm(\mathbf{c}(\gamma(M, N)); z)$-eigenfunctions with eigenvalues $2\cosh(a_\mp y)$, $\mathrm{Re}\, y \geqslant \Lambda_{M,N}$, is spanned by the two functions $F^{(\pm)}_{M,N}(z, y)$ (recall the paragraph containing (1.9)).

3. Shifting parameters in the *R*-function

Let us now trade the spectral parameter y for the spectral parameter $\hat{v}$ occurring in the R-function by setting

$$y = \alpha \hat{v} \qquad \alpha = 2\pi/a_+a_-. \tag{3.1}$$

This entails that the joint $A_\delta(\mathbf{c}(\gamma(M, N)); v)$-eigenvalues $2\cosh(a_{-\delta} y)$ of $F^{(\pm)}_{M,N}(v, y)$ and the joint eigenvalues $2c_\delta(2\hat{v})$ of $R_r(\gamma(M, N); v, \hat{v})$ become equal. Fixing at first $a_+/a_- \notin \mathbb{Q}$, we deduce from (2.32), (2.24) and evenness of R in v that for $\mathrm{Re}(\alpha\hat{v}) \geqslant \Lambda_{M,N}$ there exists a proportionality factor $p_{M,N}(\hat{v})$ such that

$$R_r(\gamma(M, N); v, \hat{v}) = p_{M,N}(\hat{v})\big(F^{(+)}_{M,N}(v, \alpha\hat{v}) + F^{(-)}_{M,N}(v, \alpha\hat{v})\big). \tag{3.2}$$

We now exploit (3.2) to establish the action of the above shifts on R_r. We begin by focusing on $S_+^{(r_\mu)}$. Assuming $M_\mu \geqslant 0$, we have by construction

$$S_+^{(r_\mu)}(\gamma(M, N); v) F^{(\delta)}_{M,N}(v, \alpha\hat{v}) = F^{(\delta)}_{M+e_\mu,N}(v, \alpha\hat{v}). \tag{3.3}$$

Thus we infer from (3.2) and its version for $M \to M + e_\mu$ that for $\mathrm{Re}(\alpha\hat{v})$ greater than $\max(\Lambda_{M,N}, \Lambda_{M+e_\mu,N})$ we have

$$S_+^{(r_\mu)}(\gamma(M, N); v) R_r(\gamma(M, N); v, \hat{v}) = \frac{p_{M,N}(\hat{v})}{p_{M+e_\mu,N}(\hat{v})} R_r(\gamma(M + e_\mu, N); v, \hat{v}). \tag{3.4}$$

Taking $\text{Re}\, v \to \infty$, we can easily calculate the dominant asymptotics of lhs and rhs from (2.2), (2.13) and (1.52), (1.53). Comparing the results and using the ($\delta = -$version of the) G-AΔEs [13]

$$\frac{G(a_+, a_-; z + ia_\delta/2)}{G(a_+, a_-; z - ia_\delta/2)} = 2c_{-\delta}(z) \tag{3.5}$$

we obtain

$$\frac{p_{M,N}(\hat{v})}{p_{M+e_\mu,N}(\hat{v})} = 2c_+(2\hat{v}) + 2c_+(2i(J\gamma(M,N))_\mu + ia_-) \qquad M_\mu \geqslant 0. \tag{3.6}$$

Consider next the two functions

$$F_1(\gamma; v, \hat{v}) \equiv S_+^{(r_\mu)}(\gamma; v) R_r(\gamma; v, \hat{v}) \tag{3.7}$$

$$F_2(\gamma; v, \hat{v}) \equiv [2c_+(2\hat{v}) + 2c_+(2i\hat{\gamma}_\mu + ia_-)] R_r(\gamma + a_- r_\mu; v, \hat{v}) \tag{3.8}$$

for parameters in Π (1.34) and complex variables v and $\hat{v}$. Both functions are real-analytic in the parameters and meromorphic in v and $\hat{v}$. Choosing $a_+/a_- \notin \mathbb{Q}$ and $M, N \in \mathbb{Z}^4$ with $M_\mu \geqslant 0$, they are equal for $\gamma = \gamma(M, N)$ and $\text{Re}\, \hat{v}$ sufficiently large (as follows from the previous paragraph). Hence they are equal for all $\hat{v} \in \mathbb{C}$. Now the $\gamma(M, N)$ with $M_\mu \geqslant 0$ are *dense* in $\mathbb{R}^4$. (This can be quickly checked by observing first

$$J\gamma(M,N) = \gamma_f + \sum_{\nu=0}^{3} (M_\nu a_- + N_\nu a_+) e_\nu. \tag{3.9}$$

Next, note that the numbers $ka_- + la_+$, with $l \in \mathbb{Z}$ and $k \in \mathbb{Z}$ or $k \in \mathbb{N}$, are dense in $\mathbb{R}$, since a_+/a_- is irrational.) Therefore, equality follows for all $\gamma \in \mathbb{R}^4$. Finally, the numbers $(a_+, a_-) \in (0, \infty)^2$ with a_+/a_- irrational are dense in $(0, \infty)^2$. Hence equality of F_1 and F_2 follows for arbitrary parameters and variables.

This reasoning can be repeated for $S_-^{(r_\mu)}(\gamma; v)$, yielding as the analogue of (3.6)

$$\frac{p_{M,N}(\hat{v})}{p_{M,N+e_\mu}(\hat{v})} = 2c_-(2\hat{v}) + 2c_-(2i(J\gamma(M,N))_\mu + ia_+) \qquad N_\mu \geqslant 0. \tag{3.10}$$

The upshot is that we have proved the shift relations

$$S_\delta^{(r_\mu)}(\gamma; v) R_r(\gamma; v, \hat{v}) = [2c_\delta(2\hat{v}) + 2c_\delta(2i\hat{\gamma}_\mu + ia_{-\delta})] R_r(\gamma + a_{-\delta} r_\mu; v, \hat{v}). \tag{3.11}$$

Considering next $S_\delta^{(-r_\mu)}(\gamma; v)$, a simplification occurs: upon comparing asymptotics and using (3.5), we find

$$\frac{p_{M,N}(\hat{v})}{p_{M-e_\mu,N}(\hat{v})} = 1 \qquad M_\mu \leqslant 0 \qquad \frac{p_{M,N}(\hat{v})}{p_{M,N-e_\mu}(\hat{v})} = 1 \qquad N_\mu \leqslant 0. \tag{3.12}$$

Thus the above argument yields

$$S_\delta^{(-r_\mu)}(\gamma; v) R_r(\gamma; v, \hat{v}) = R_r(\gamma - a_{-\delta} r_\mu; v, \hat{v}). \tag{3.13}$$

(As a check, observe that the result of combining (3.11) and (3.13) is consistent with (2.16).)

As we have already seen in the introduction, we have $p_{0,0}(\hat{v}) = 1$, cf (1.54) and (3.2). On account of the recurrence (3.12), this entails

$$p_{M,N}(\hat{v}) = 1 \qquad M, N \in (-\mathbb{N})^4. \tag{3.14}$$

More generally, the proportionality factors $p_{M,N}(\hat{v})$, $M, N \in \mathbb{Z}^4$, can be calculated recursively by using also (3.6) and (3.10), yielding a hyperbolic function (in the sense defined below

(1.57)). Thus $R_r(\gamma; v, \hat{v})$ is elementary for all γ of the form $\gamma(M, N)$ (2.19), just as $F^{(\pm)}_{M,N}(v, \alpha\hat{v})$. (Indeed, the general term (2.26) with $y \to \alpha\hat{v}$ is hyperbolic.)

To proceed, we exploit the self-duality relation (1.43). Combined with (3.11) and (3.13), it entails

$$S^{(r_\mu)}_\delta(\hat{\gamma}; \hat{v})R_r(\gamma; v, \hat{v}) = [2c_\delta(2v) + 2c_\delta(2\mathrm{i}\gamma_\mu + \mathrm{i}a_{-\delta})]R_r(\gamma + a_{-\delta}e_\mu; v, \hat{v}) \tag{3.15}$$

$$S^{(-r_\mu)}_\delta(\hat{\gamma}; \hat{v})R_r(\gamma; v, \hat{v}) = R_r(\gamma - a_{-\delta}e_\mu; v, \hat{v}). \tag{3.16}$$

Now in the appendix we have seen that the linear combinations of e_μ and r_μ, with $\mu = 0, 1, 2, 3$, give rise to the D_4 weight lattice P. Thus it easily follows that $R_r(\gamma; v, \hat{v})$ is elementary for all γ in the set $\gamma_f + a_-P_- + a_+P_+$, where $P_\pm$ are copies of P.

In order to show that R_r is elementary on the larger set Π_{el} (1.56), we first derive an alternative representation for Π_{el}. To this end we define a subset $\mathcal{Z}$ of $\mathbb{Z}^4 \times \mathbb{Z}^4$ by requiring that for $(M, N) \in \mathcal{Z}$ the four pairs $(M_\mu, N_\mu), \mu \in \{0, 1, 2, 3\}$, are distinct mod(2); equivalently, the pairs are of the form (even, even), (odd, odd), (even, odd), (odd, even). We now claim that Π_{el} can be rewritten as

$$\Pi_{\mathrm{el}} = \left\{ (a_+a_-, \gamma) \in \Pi \mid \gamma = \frac{1}{2}\sum_{\nu=0}^{3}(M_\nu a_- + N_\nu a_+)e_\nu,\ (M, N) \in \mathcal{Z} \right\}. \tag{3.17}$$

To prove this claim, we denote the set on the rhs by $\mathcal{R}$. Clearly, $\mathcal{R}$ contains all W-transforms of γ_f (1.33). Adding multiples of $a_\delta r_\mu$ and $a_\delta e_\mu$ to $w(\gamma_f)$, we stay in $\mathcal{R}$, so that $\Pi_{\mathrm{el}} \subset \mathcal{R}$. On the other hand, for $(a_+, a_-, \gamma) \in \mathcal{R}$ we need only add suitable multiples of $a_\delta e_\nu$ to γ to obtain a permutation of γ_f. Hence we have $\mathcal{R} \subset \Pi_{\mathrm{el}}$, and so (3.17) follows.

The crux is now that $c(a_+, a_-, \gamma; z)$ is a rational function of $e_+(z)$ and $e_-(z)$ for parameters in Π_{el}. This is not obvious from the definition (1.56), but it readily follows from (3.17). Indeed, due to the AΔ Es (3.5), the functions

$$\frac{G(a_+, a_-; z + \mathrm{i}ka_+ + \mathrm{i}la_-)}{G(a_+, a_-; z)} \qquad k, l \in \mathbb{Z} \tag{3.18}$$

are rational functions of $e_\pm(z)$, so by the duplication formula (1.55) and the representation (3.17), $c(a_+, a_-, \gamma; z)$ is a product of four functions of the form (3.18), with $z \to z + \mathrm{i}(a_+ + a_-)/2, z + \mathrm{i}a_-/2, z + \mathrm{i}a_+/2, z$.

From (1.48) it is now plain why R_r is elementary on Π_{el}: when we choose $\gamma^{(2)} = \gamma_f + a_-\lambda_- + a_+\lambda_+$ and $\gamma^{(1)} = w(\gamma^{(2)})$ in (1.48), then we obtain a hyperbolic rhs, so elementarity of $R_r(\gamma^{(2)}; v, \hat{v})$ entails elementarity of $R_r(\gamma^{(1)}; v, \hat{v})$ (recall the paragraph containing (1.57)). Moreover, since the c-function factors in (1.47) are hyperbolic on Π_{el} (as follows from the previous paragraph), we also obtain elementarity of $\mathcal{E}(\gamma; v, \hat{v})$ for parameters in Π_{el}.

Due to the uniqueness of the representation (1.57), features of the functions R_r and $\mathcal{E}$ for parameters in Π imply corresponding features of the summands (1.58) on the dense subset Π_{el}. In particular, $R^{(\delta)}_r$ satisfies (1.42)–(1.44), whereas (1.45) gives rise to

$$R^{(+)}_r(-v, \hat{v}) = R^{(+)}_r(v, -\hat{v}) = R^{(-)}_r(v, \hat{v}). \tag{3.19}$$

The functions $R^{(\pm)}_r$ also obey the shift relations (3.11), (3.13), (3.15) and (3.16), and they are joint eigenfunctions of the same four AΔOs as R_r. Likewise, $\mathcal{E}^{(\pm)}$ are D_4-invariant joint eigenfunctions of the four similarity transformed AΔOs, with asymptotics

$$\mathcal{E}^{(+)}(\gamma; v, \hat{v}) \sim \exp(\mathrm{i}\alpha v\hat{v}) \qquad \mathcal{E}^{(-)}(\gamma; v, \hat{v}) \sim \frac{c(\hat{\gamma}; -\hat{v})}{c(\hat{\gamma}; \hat{v})}\exp(-\mathrm{i}\alpha v\hat{v}) \qquad \mathrm{Re}\, v \to \infty \tag{3.20}$$

cf (1.50) and (1.51). (Note that the similarity transformed parameter shifts have hyperbolic coefficients for parameters in Π_{el}, but not for parameters in $\Pi \setminus \Pi_{el}$.)

In view of (3.2) and (3.14), we have

$$R_r^{(\sigma)}(\gamma(M,N); v, \hat{v}) = F_{M,N}^{(\sigma)}(v, \alpha\hat{v}) \qquad \sigma = +,- \qquad M, N \in (-\mathbb{N})^4. \tag{3.21}$$

For other $\gamma(M,N)$, the relation between $R_r^{(\sigma)}(\gamma(M,N); v, \hat{v})$ and the auxiliary functions $F_{M,N}^{(\sigma)}(z,y)$ can be in principle obtained from (3.2) by using the recurrence relations (3.6), (3.10) and (3.12).

To conclude, we point out that for $a_+/a_- \in \mathbb{Q}$ there are infinitely many distinct pairs $(M,N) \in \mathbb{Z}^4$ yielding the same $\gamma(M,N) \in \mathbb{R}^4$ (this can be seen from (2.19)). Now it is evident that $R_r^{(\sigma)}(\gamma(M,N); v, \hat{v})$ is the same for all pairs. Taking (3.21) into account, one might guess that the auxiliary functions $F_{M,N}^{(\sigma)}(z,y)$ coincide as well. In general this is false, however. A simple example is the case $a_+ = a_-$. Here we have $\gamma(M,-M) = \gamma_f$ for all $M \in \mathbb{Z}^4$, but $F_{M,-M}^{(\sigma)}(z,y)$ clearly depends on M.

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Appendix. Some D_4 features

In this appendix we collect some well-known material connected to the D_4 Lie algebra, cf e.g. [14, 15]. Our notational conventions agree with the applications in the main text. The D_4 root system lives in $\mathbb{R}^4$, whose canonical basis we denote by e_μ, $\mu = 0,1,2,3$. It is given by the vectors

$$\delta e_\mu + \delta' e_\nu \qquad \delta, \delta' = +,- \qquad \mu, \nu = 0,1,2,3 \qquad \mu \neq \nu \tag{A.1}$$

and its Weyl group W is the product of the permutation group S_4 and the group of 'even' sign changes, i.e.,

$$(p_0, p_1, p_2, p_3) \mapsto (\delta_0 p_0, \delta_1 p_1, \delta_2 p_2, \delta_3 p_3) \qquad \delta_\mu \in \{\pm 1\} \qquad \sum_{\mu=0}^{3} \delta_\mu \in \{0, \pm 4\}. \tag{A.2}$$

The root lattice Q is generated by the simple roots

$$e_0 + e_1 \qquad e_1 + e_2 \qquad e_2 + e_3 \qquad e_2 - e_3. \tag{A.3}$$

The weight lattice P consists of all $\lambda \in \mathbb{R}^4$ satisfying $(\lambda, \alpha) \in \mathbb{Z}$, $\alpha \in Q$. It is generated by the fundamental weights

$$\lambda_0 \equiv e_0 \qquad \lambda_1 \equiv e_0 + e_1 \qquad \lambda_2 \equiv (e_0 + e_1 + e_2 + e_3)/2 \qquad \lambda_3 \equiv (e_0 + e_1 + e_2 - e_3)/2 \tag{A.4}$$

which are the highest weights of the defining, adjoint, even spinor and odd spinor representation, respectively.

For the above parameter shifts, the eight weights of the defining and even spinor representation are the relevant ones. The former are $\pm e_\mu$, $\mu = 0,1,2,3$, while we find it convenient to denote the latter by $\pm r_\mu$, where

$$\begin{aligned} r_0 &\equiv (1,1,1,1)/2 \qquad & r_1 &\equiv (1,1,-1,-1)/2 \\ r_2 &\equiv (1,-1,1,-1)/2 \qquad & r_3 &\equiv (1,-1,-1,1)/2. \end{aligned} \tag{A.5}$$

Indeed, the vectors r_μ are the rows of the matrix J (1.38), which plays a crucial role. Clearly, it satisfies

$$Jr_\mu = e_\mu \qquad Je_\mu = r_\mu \qquad \mu = 0, 1, 2, 3 \tag{A.6}$$

$$JWJ = W. \tag{A.7}$$

Finally, we note that the sublattice of P generated by r_μ and e_μ, $\mu = 0, 1, 2, 3$, contains the fundamental weights (A.4), so that it equals P.

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